FIELDS OF RANDOMLY DISTRIBUTED DISLOCATIONS AND FORCE DIPOLES IN AN INFINITE ELASTIC ANISOTROPIC MEDIUM

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A uniform mathematical representation of singularities of the stress field of an elastic medium is proposed. Characteristics of one-, two- and three-dimensional distribution of dislocations and force dipoles are introduced, and a general method of calculating the corresponding stresses in an infinite elastic anisotropic medium is discussed. The equivalence of the stress fields of dislocations and force dipoles is demonstrated, and the application of the method is illustrated on some simple problems of spherical and cylindrical symmetry.

1. **Delta functions defined by surfaces.** * Let K denote a fundamental function space of Euclidean space R_3 consisting of infinitely differentiable finite functions $\varphi(x)$ of a point x (x^1, x^2, x^3), and let L be a certain curve. The equation

$$\int \delta(L)\varphi(x)\,dx = \int\limits_L \varphi(x_L)\,dL, \qquad (1.1)$$

where dL is a line element, defines on K the generalized function $\delta(L)$. Functions $\delta(S)$ and $\delta(V)$ for a surface S and volume V are similarly defined:

$$\int \delta(S)\varphi(x)\,dx = \int\limits_S \varphi(x_S)\,dS,$$

$$\int \delta(V)\varphi(x)\,dx = \int\limits_V \varphi(x_V)\,dV, \qquad (1.2)$$

it being evident that $\delta(V)$ coincides with the characteristic function of the domain V. With respect to L, S, and V, the existence of only the right sides in (1.1) and (1.2) is assumed. It is a valid proposition that

$$\delta(L) = \int\limits_L \delta(x - x_L)\,dL, \qquad (1.3)$$

which should be understood in the sense that

$$\int dx\,\varphi(x) \int\limits_L \delta(x - x_L)\,dL = $$

$$= \int\limits_L dL \int \delta(x - x_L)\varphi(x)\,dx = \int\limits_L \varphi(x_L)\,dL.$$

Similiarly,

$$\delta(S) = \int\limits_S \delta(x - x_S)\,dS, \qquad \delta(V) = \int\limits_V \delta(x - x_V)\,dV. \quad (1.4)$$

Let, for instance, L represent the axis x^3 and S the plane x^1x^2. Then $\delta(L)$ and $\delta(S)$ may be represented as the direct product of one-dimensional δ-functions and functions identically equal to unity,

$$\delta(L) = \delta(x^1) \times \delta(x^2) \times 1(x^3),$$

$$\delta(S) = 1(x^1, x^2) \times \delta(x^3).$$

We have

$$\int\limits_S \delta(L)\,dS = \int \delta(x^1)\times\delta(x^2)\times 1(0)\,dx^1dx^2 = 1,$$

$$\int\limits_L \delta(S)\,dL = \int 1(0, 0)\times\delta(x^3)\,dx^3 = 1.$$

Hence in this case it may be stated that

$$\delta(L)\,\delta(S) = \delta(x^1) \times \delta(x^2) \times \delta(x^3) = \delta(x). \qquad (1.5)$$

It is easily seen that (1.5) does not depend on the specific form of L and S provided that they intersect at one point x = 0. Consequently, in the general case, if L intersects S at one point $x = x_0$, we have

$$\delta(L)\,\delta(S) = \delta(x - x_0). \qquad (1.6)$$

For bounded L, S, and V,

$$\int \delta(L)\,dx = l, \qquad \int \delta(S)\,dx = s, \qquad \int \delta(V)\,dx = v, \quad (1.7)$$

where l, s and v denote, respectively, length, surface area and volume.

If V contracts to a point x_0, then

$$\delta(V) \approx v\delta(x - x_0), \qquad \lim_{V\to x_0} \frac{1}{v}\delta(V) = \delta(x - x_0). \quad (1.8)$$

Similar asymptotic formulas are obtained for $\delta(L)$ and $\delta(S)$.

In addition to scalar functions we shall consider vector δ-functions $\delta(L^\nu)$ and $\delta(S^\nu)$ which, for oriented piecewise smooth L and S, are defined by relations

$$\int \delta(L^\nu)\varphi(x)\,dx = \int\limits_L \varphi(x_L)\,dL^\nu \qquad (dL^\nu = \lambda^\nu dL),$$

$$\int \delta(S^\nu)\varphi(x)\,dx = \int\limits_S \varphi(x_S)\,dS^\nu \qquad (dS^\nu = n^\nu dS), \quad (1.9)$$

where $\lambda(x_L)$ and $n^\nu(x_S)$ are, respectively, the unit vector tangent to L and the normal to S. For bounded L and S

$$\int \delta(L^\nu)\,dx = l^\nu, \qquad \int \delta(S^\nu)\,dx = s^\nu, \qquad (1.10)$$

where l^ν and s^ν are vector length and surface area. For closed L and S, these are equal to zero. If L and S intersect at one point $x = x_0$, then

$$\delta(L_\nu)\,\delta(S^\nu) = \pm\,\delta(x - x_0), \qquad (1.11)$$

depending on whether or not their orientations are

*With reference to generalized functions associated with surfaces see also [1]. It should be pointed out, however, that the functionals considered there differ from the δ-functions introduced below.

matched. In fact, by deforming L and S one can transform them into a straight line and plane with constant unit vectors λ^ν and n^ν, and reduce the problem to (1.6).

The derivatives of δ-functions are determined in the usual way, by switching the operation to the fundamental function, e.g.,

$$\int \delta_{,\mu}(S^\nu)\varphi(x)\,dx = -\int_S \varphi_{,\mu}(x_S)\,dS^\nu.$$

The above-introduced δ-functions have a simple geometrical interpretation. It can be shown that, correct to the constant factor, they constitute kernels of the corresponding averaging operators with respect to L, S and V.

Of considerable importance are relations of the Stokes formula type for δ-functions. Let V be a domain in R_3 with a piecewise smooth boundary S (with matched orientations). Then for the tensor p (indices are omitted) we have

$$\int \delta_{,\lambda}(V)\,p(x)\,dx = -\int_V \partial_\lambda p(x_V)\,dV =$$

$$= -\int_S p(x_S)\,dS_\lambda = -\int \delta(S_\lambda)\,p(x)\,dx.$$

Hence

$$\operatorname{grad} \delta(V) = -\delta(S). \qquad (1.12)$$

Consequently, for a closed surface S,

$$\operatorname{rot} \delta(S) = 0. \qquad (1.13)$$

Let now S be a two-dimensional surface with a piecewise smooth boundary L (with matched orientations). Then, from the formulas ($\varepsilon^{\lambda\mu\nu}$ is an antisymmetric pseudotensor)

$$\int \varepsilon^{\lambda\mu\nu}\delta_{,\mu}(S_\lambda)\,p(x)\,dx = -\int_S \varepsilon^{\lambda\mu\nu}\partial_\mu p(x_S)\,dS_\lambda =$$

$$= -\int_L p(x_L)\,dL^\nu = -\int \delta(L^\nu)\,p(x)\,dx$$

there follows

$$\operatorname{rot} \delta(S) = \delta(L). \qquad (1.14)$$

Hence, for a closed contour

$$\operatorname{div} \delta(L) = 0. \qquad (1.15)$$

Now let there be given an oriented contour L and on this contour a certain scalar or tensor function $f(x_L)$. Then, the corresponding δ-function with weight $f(x_L)$ is given by

$$\int [f(x_L)\,\delta(L)]\,\varphi(x)\,dx = \int_L \varphi(x_L)\,f(x_L)\,dL. \qquad (1.16)$$

The other weighted δ-functions are determined in a similar way.

The derivatives of the weighted δ-functions are determined in the usual way, by switching the operation to $\varphi(x)$. When this cannot lead to misunderstanding, brackets in the expressions for the weighted δ-functions will be omitted, e.g., $\operatorname{rot}^{\mu\lambda} M_\lambda(x_L)\,\delta(L)$, where the operator rot applies to the whole expression. It is easily seen that the following relations hold:

$$\delta(L^\nu) = \lambda^\nu(x_L)\,\delta(L), \qquad \delta(S^\nu) = n^\nu(x_S)\,\delta(S), \qquad (1.17)$$

where λ^ν and n^ν are, respectively, the unit tangent vector and normal.

2. **Characteristics of the distribution of dislocations and force dipoles.** The Kröner equation, which relates internal strains ε with the overall density of internal stress sources (incompatibility) η, takes the form [2]

$$\operatorname{Rot} \varepsilon = \eta, \qquad (2.1)$$

where the operator Rot is defined by

$$\operatorname{Rot}^{\lambda\mu\rho\nu} = \varepsilon^{\lambda\tau\rho}\varepsilon^{\mu\sigma\nu}\partial_\tau\partial_\sigma \qquad \left(\partial_\tau = \frac{\partial}{\partial x^\tau}\right).$$

It is assumed now that the internal stresses are produced by dislocations. If, as is usual, we introduce a dislocation density tensor α which satisfies the condition div α = 0, then, as is known,* for η we have

$$\eta^{\lambda\mu} = \operatorname{rot}^{(\lambda}_{\cdot\nu}\alpha^{\mu)\nu}. \qquad (2.2)$$

It should be pointed out that unlike η, in the framework of the continuous model α is a directly measurable value.

In addition to η and α, we shall introduce a third (perhaps most convenient in application) characteristic of the distribution of dislocations. Let us assume that

$$\alpha = \operatorname{rot} \mu, \qquad (2.3)$$

where μ, denoting the tensor of the density of dislocation moments, is a quantity which can be determined correct to the gradient of an arbitrary vector field. We then have

$$\eta^{\lambda\mu} = \operatorname{rot}^{(\lambda}_{\cdot\nu}\alpha^{\mu)\nu} = \operatorname{rot}^{(\lambda}_{\cdot\nu}\operatorname{rot}^{\mu)}_{\cdot\rho}\mu^{\rho\nu} = \operatorname{Rot}^{\lambda\mu}_{\cdot\cdot\rho\nu}\mu^{(\rho\nu)}. \qquad (2.4)$$

Hence it follows that η and, consequently, ε (but not α) are not affected by the antisymmetric component μ.

Let us consider a surface S bounded by a contour L, and let b^ν denote a constant vector. In view of (1.15), the expression

$$\alpha^{\mu\nu}(x) = b^\nu\delta(L^\mu) \qquad (2.5)$$

may be regarded as the density of some distribution of dislocations. Let us now choose an arbitrary surface F intersecting L at one point, and let us consider the flux α through F. Taking into account (1.11), we find

$$\int_F \alpha^{\mu\nu}(x_F)\,dF_\mu = b^\nu \int_F \delta(L^\mu)\,dF_\mu = \pm b^\nu.$$

Here the signs plus or minus are used depending

*Here and henceforth, round brackets denote symmetrization with respect to those indices.

on whether or not the orinetations of L and F are matched.

Since this flux does not depend on the choice of F intersecting L, and since it is equal to zero if F does not intersect L, it follows that (2.4) represents the density corresponding to an edge dislocation with contour L and Burgers vector b^ν. On the other hand, taking into account (1.14) and (1.17), we have

$$\alpha^{\mu\nu}(x) = \mathrm{rot}^{\mu}_{\cdot\rho} b^\nu \delta(S^\rho) = \mathrm{rot}^{\mu}_{\cdot\rho} b^\nu n^\rho (x_S)\,\delta(S), \qquad (2.6)$$

i. e., α may also be interpreted as the distribution of dislocations over S with constant Burgers vector b^ν and a surface density of moments $M^{\rho\nu}(x_S) = b^\nu n^\rho(x_S)$. The quantity

$$\mu^{\rho\nu}(x) = b^\nu \delta(S^\rho) = M^{\rho\nu}(x_S)\,\delta(S) \qquad (2.7)$$

is the density of dislocation moments corresponding to the distributions (2.6) or (2.4). This may be extended to the case of an arbitrary distribution of dislocations. For instance, let the dislocations be distributed in some domain V. Then the most general characteristics of the dislocation distribution are the densities

$$\mu^{\rho\nu}(x) = M^{\rho\nu}(x_V)\,\delta(V),$$
$$\alpha^{\mu\nu}(x) = \mathrm{rot}^{\mu}_{\cdot\rho} M^{\rho\nu}(x_V)\,\delta(V),$$
$$\eta^{\lambda\mu}(x) = \mathrm{Rot}^{\lambda\mu}_{\cdot\cdot\rho\nu} M^{(\rho\nu)}(x_V)\,\delta(V), \qquad (2.8)$$

where $M^{\rho\nu}(x_V)$ is the volume density of dislocation moments.

In the case of dislocations distributed over a surface S or contour L, $\delta(V)$ should be replaced by $\delta(S)$ or $\delta(L)$ and $M^{\rho\nu}(x_V)$ by $M^{\rho\nu}(x_S)$ or $M^{\rho\nu}(x_L)$, i. e., by surface or line densities of dislocation moments.

In the limiting case of an elementary dislocation, which can be obtained, for instance, by contracting the surface S in (2.6) to a point x_0, or—what amounts to the same—by removing the observation point to a sufficient distance x, the expressions for μ, α, and η become [taking into account (1.8)]

$$\mu^{\rho\nu}(x) = M^{\rho\nu}\delta(x - x_0), \qquad \alpha^{\mu\nu}(x) = \mathrm{rot}^{\mu}_{\cdot\rho} M^{\rho\nu}\delta(x - x_0),$$
$$\dot{\eta}^{\lambda\mu}(x) = \mathrm{Rot}^{\lambda\mu}_{\cdot\cdot\rho\nu} M^{(\rho\nu)}\,\delta(x - x_0), \qquad M^{\rho\nu} = b^\nu s^\rho, \qquad (2.9)$$

where s^ρ is the vector area of S.

The distribution of force dipoles in V is similarly characterized by

$$f^\rho(x) = -\partial_\nu Q^{\rho\nu}(x_V)\,\delta(V), \qquad q^{\rho\nu}(x) = Q^{\rho\nu}(x_V)\,\delta(V), \qquad (2.10)$$

where $f^\rho(x)$ is the density of the equivalent body force, $q^{\rho\nu}(x)$ the density of dipole moments, and $Q^{\rho\nu}(x_V)$ the volume density of dipole moments. To obtain the distribution of dipoles on S or L, one should replace $\delta(V)$ by $\delta(S)$ or $\delta(L)$ and $Q^{\rho\nu}(x_V)$ by $Q^{\rho\nu}(x_S)$ or $Q^{\rho\nu}(x_L)$, i.e., by the surface or line density of dipole moments.

Let us compare the different characteristics of the distribution of dislocations and force dipoles, assuming that the other characteristics of the medium are constant. The dislocation density α gives full information about dislocations regarded as a physical object, but from the standpoint of internal stresses the information it gives is useless. The dislocation "incompatibility" η contains complete information about internal stresses produced by dislocations, but not about dislocations as such. The moment density μ contains complete information about dislocations, but correct only to $\nabla \mathbf{v}$, where $\mathbf{v}$ is an arbitrary vector; in relation to internal stresses μ is defined correct to def $\mathbf{v} + \omega$, where ω is an arbitrary antisymmetrical tensor.*

The equivalent density of body forces f contains complete information about force dipoles as sources o of stresses but not about the dipole distribution. The dipole moment density q contains complete information relating to dipole distribution; however, with respect to stresses it is defined correct to an arbitrary tensor p which satisfies the condition $p \cdot \Delta = 0$.

3. **Fields of dislocations and force dipoles.** It follows from (2.1) that internal stresses in the linear theory satisfy equations

$$\mathrm{Rot}\, C^{-1}\sigma = \eta, \qquad \mathrm{div}\,\sigma = 0, \qquad (3.1)$$

where C is the elastic constant tensor. If the distribution of dislocations is known and nonzero only in a finite domain, the solution of (3.1) in an infinite medium may, taking into account (2.4), be written in the form

$$\sigma^{\alpha\beta}(x) = H^{\alpha\beta}_{\cdot\cdot\lambda\mu} * \eta^{\lambda\mu} =$$
$$= \mathrm{rot}^{(\lambda}_{\cdot\nu} H^{|\alpha\beta|}_{\cdot\cdot\lambda\mu} * \alpha^{\mu)\nu} = \mathrm{Rot}^{\lambda\mu}_{\cdot\cdot\rho\nu} H^{\alpha\beta}_{\cdot\cdot\lambda\mu} * \mu^{(\rho\nu)}, \qquad (3.2)$$

where $H^{\alpha\beta}_{\cdot\cdot\lambda\mu}(x)$ is the Green's tensor for internal stresses [6]. The operation * denotes, as usual, convolution.

Let us correlate the distribution of dislocations with a moment density $\mu^{\rho\nu}$ and the distribution of force dipoles with dual symmetrical moment density $q^{\rho\nu}$, using the relation

$$q^{\rho\nu} = -C^{\rho\nu}_{\cdot\cdot\sigma\tau}\mu^{\sigma\tau}, \qquad \mu^{(\rho\nu)} = -C^{-1\rho\nu}_{\cdot\cdot\sigma\tau} q^{\sigma\tau}. \qquad (3.3)$$

Then, stresses σ' corresponding to q satisfy equations

$$\mathrm{Rot}\, C^{-1}\sigma' = 0, \qquad \mathrm{div}\,\sigma' = -f \quad (f = -\mathrm{div}\, q), \qquad (3.4)$$

whose solution may be written (with the aid of the

*It can be shown that μ formally coincides with Kröner's plastic distortion β^P [2], Indenbom's residual distortion ε° [3] and Kroupa's dislocation loop density γ [4] (see also [7]). Generally speaking, however, all these parameters are not functions of state and depend on the history of the medium [4]. If therefore the discussion is confined, as in the present case, within the framework of the elastic continuum, it is advisable not to ascribe to μ any independent physical meaning but to regard it as a tensor potential of the dislocation density $\boldsymbol{\alpha}$.

Green's stress tensor of the theory of elasticity) in the form

$$\sigma^{\alpha\beta}(x) = G^{\alpha\beta}_{\cdot\cdot\rho} * f^{\rho}. \qquad (3.5)$$

Equating (3.4) to (3.1) leads directly to an important relation:

$$\sigma' = \sigma' - q = \sigma' + C\mu. \qquad (3.6)$$

In other words, each problem of the (linear) continuous theory of dislocations for an infinite medium with a given moment density μ can be unambiguously correlated with a problem of the classical theory of elasticity with a dual density of force dipole moments q, the corresponding stresses being related by (3.6), which were first written in explicit form (for an isotropic medium) in [4].

If (3.5) is transformed (taking into account (3.3)) for σ', we obtain an expression for the displacement u_λ corresponding to the force dipole density

$$u_\lambda(x) = G^{\rho\nu}_{\cdot\cdot\lambda} * \mu_{\rho\nu} = U_{\lambda\rho} * f^{\rho}, \qquad (3.7)$$

where $U_{\lambda\rho}$ is the Green's displacement tensor of the theory of elasticity.

Since in the region, in which $\mu = 0$, the stress σ coincides with σ', in this (and only in this) region **u** may be regarded as a, generally speaking, nonunique vector potential, in terms of which ε and σ are (locally expressed by the usual formulas. With this reservation, **u** in this region may be interpreted as a displacement field produced by a given dislocation distribution.

Let us now consider some more important cases of the distribution of dislocations and dual moments and the corresponding stress fields.

Let $M^{\rho\nu}(x_S)$ be the surface density of dislocations distributed on a surface S with contour L, and let $Q^{\rho\nu}(x_S)$ denote the dual surface density of dipole moments. Then for the stresses we have

$$\sigma'^{\alpha\beta} = -G^{\alpha\beta}_{\cdot\cdot\rho,\nu} * Q^{\rho\nu}(x_S)\,\delta(S) = -\int_S G^{\alpha\beta}_{\cdot\cdot\rho,\nu}(x - x_{S'})\,Q^{\rho\nu}(x_S)\,dS, \qquad (3.8)$$

$$\sigma^{\alpha\beta} = \mathrm{Rot}^{\lambda\mu}_{\cdot\cdot\rho\nu} H^{\alpha\beta}_{\cdot\cdot\lambda\mu} * M^{\rho\nu}(x_S)\,\delta(S) = \int_S \mathrm{Rot}^{\lambda\mu}_{\cdot\cdot\rho\nu} H^{\alpha\beta}_{\cdot\cdot\lambda\mu}(x - x_S)\,M^{\rho\nu}(x_S)\,dS. \qquad (3.9)$$

In accordance with (3.6) their difference is equal to $Q^{\rho\nu}(x_S)\,\delta(S)$, i.e., to a singularity concentrated on S.

If $M^{\rho\nu}(x_S) = b^\nu n^\rho(x_S)$, and, consequently, $\mu^{\rho\nu}$ coincides with (2.7), this distribution corresponds to an edge dislocation with density (2.5). From (3.2) it follows that the stress field is in the form[1]

$$\sigma^{\alpha\beta} = H^{\alpha\beta}_{\cdot\cdot\lambda\mu} * \mathrm{rot}^{(\lambda|\nu|} b_\nu \delta(L^{\mu)}) = b_\nu \oint_L H^{\alpha\beta}_{\cdot\cdot\lambda\mu,\tau}(x - x_L)\,\varepsilon^{\tau\nu(\lambda}\,dL^{\mu)}. \qquad (3.10)$$

It has singularities only on the contour L. In contrast, the dual moment stress field

$$\sigma'^{\alpha\beta} = b_\lambda C^{\lambda\tau\rho\nu} \int_S G^{\alpha\beta}_{\cdot\cdot\rho,\nu}(x - x_S)\,dS_\tau \qquad (3.11)$$

has a singularity on S; it must not be represented in the form of an integral over L.

Substituting (2.7) in (3.7), we find for an edge dislocation

$$u_\lambda = b_\nu \int_S G^{\nu\mu}_{\cdot\cdot\lambda}(x - x_S)\,dS_\mu. \qquad (3.12)$$

For an isotropic medium, this formula becomes the Burgers formula, which is usually regarded as an expression for the displacements of an edge dislocation. (More accurately, they should be regarded as displacements corresponding to the dual density of force dipoles distributed on S.)

Let us now consider dislocations distributed in domain V with a boundary S and a constant volume moment density $M^{\rho\nu}(x_V)$. The dislocation density, taking into account (1.12), may be represented in the form[2]

$$\alpha^{\mu\nu}(x) = \mathrm{rot}^{\mu}_{\cdot\rho} M^{\rho\nu}\delta(V) = \varepsilon^{\mu\rho\tau} M_\rho^{\cdot\nu}\delta(S_\tau), \qquad (3.13)$$

i.e., this distribution is exactly equivalent to the corresponding distribution of uncompensated dislocations on S, and stresses inside and outside S are given by (3.9). It should be pointed out that if q or μ satisfy conditions div q = 0 or Rot $\mu = 0$, then in these cases, respectively, $\sigma = -q$, $\sigma' = 0$ or $\sigma = 0$, $\sigma' = q$.

4. Problems of spherical and cylindrical symmetry. To illustrate the use of the general relations, we shall analyze some simple problems which can be solved by a direct method.[3]

The case of spherical symmetry will be discussed first. Let the domain V be a sphere bounded by a spherical surface S of radius R. It can easily be seen that in spherical coordinates r, ϑ, φ

$$\delta(V) = \theta(R - r), \qquad \delta(S) = \delta(r - R), \qquad (4.1)$$

where $\theta(r) = 1$ for $r > 0$ and $\theta(r) = 0$ for $r < 0$.

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[1] A representation of σ in terms of an integral over L was obtained for an isotropic medium by Peach and Koehler [10].

[2] This case corresponds to the problem of an inclusion analyzed by J. Eshelby [9].

[3] The more general problem of an ellipsoidal inclusion is analyzed in [9].

For the sake of simplicity we shall consider only tensors with nonzero components $A_{rr} \equiv A_r$, $A_{\vartheta\vartheta} = A_{\varphi\varphi} \equiv A_\vartheta$. Then, the components of the dislocation incompatibility $\eta(r)$ given by (2.4) will, in spherical coordinates, be written in the form [10]

$$\eta_r = \frac{2}{r}\left[\mu'_\vartheta + \frac{1}{r}(\mu_\vartheta - \mu_r)\right], \qquad \eta_\vartheta = \frac{1}{r}\left[(r\mu_\vartheta)'' - \mu_r'\right]. \quad (4.2)$$

It can readily be verified that the condition div $\eta = 0$, or

$$\eta_r' + 2r^{-1}(\eta_r - \eta_\vartheta) = 0 \quad (4.3)$$

is identically satisfied. Consequently, η has one significant component η_r, in terms of which η_ϑ may be expressed with the aid of (4.3).

In our case the set of equations (3.1) for the internal stresses for an isotropic medium assumes the form (λ, μ are Lamé coefficients)

$$\frac{2}{r}\varepsilon_\vartheta' + \frac{2}{r^2}(\varepsilon_\vartheta - \varepsilon_r) = \eta_r, \qquad \sigma_r' + \frac{2}{r}(\sigma_r - \sigma_\vartheta) = 0,$$
$$\sigma_r = (\lambda + 2\mu)\varepsilon_r + 2\lambda\varepsilon_\vartheta, \qquad \sigma_\vartheta = \lambda\varepsilon_r + 2(\lambda + \mu)\varepsilon_\vartheta. \quad (4.4)$$

Let there be on a sphere S a distribution of dislocations with Burgers vector oriented along the radius and equal to b = const. Then from the general formula

$$\mu^{\rho\nu}(x) = b^\nu n^\rho \delta(S), \quad (4.5)$$

taking into account (4.1), we find for the moment density

$$\mu_r = b\delta(r - R), \qquad \mu_\vartheta = 0. \quad (4.6)$$

Substituting in (4.2), we obtain an expression for the significant component

$$\eta_r = -\frac{2b}{R^2}\delta(r - R). \quad (4.7)$$

The solution of (4.4) is in the form

$$\sigma_r = \frac{4\mu(3\lambda + 2\mu)}{3(\lambda + 2\mu)}\frac{b}{R}\left[\theta(R - r) + \frac{R^3}{r^3}\theta(r - R)\right],$$
$$\sigma_\vartheta = \frac{4\mu(3\lambda + 2\mu)}{3(\lambda + 2\mu)}\frac{b}{R}\left[\theta(R - r) - \frac{R^3}{2r^3}\theta(r - R)\right]. \quad (4.8)$$

A natural interpretation of this problem is as follows. Into a spherical cavity of radius R a sphere of the same material is placed with a radial clearance (or tightness) b, and the two parts are welded together. If instead of welding a force double layer is applied at the interface, then (in accordance with (3.6)) the corresponding stresses will differ by the dual moment density

$$q_r = -(\lambda + 2\mu)\, b\,\delta(r - R), \qquad q_\vartheta = -\lambda\, b\,\delta(r - R). \quad (4.9)$$

Another interpretation can be formulated if it is taken into account that a temperature distribution following an arbitrary law T(x) corresponds to an incompatibility [2]

$$\eta^{\lambda\mu}(x) = \mathrm{Rot}^{\lambda\mu}_{\cdot\cdot\rho\nu}\gamma T(x)\delta^{\rho\nu}, \quad (4.10)$$

where γ is the thermal expansion coefficient. Assuming $T(r) = T_0\theta(R - r)$, T_0 = const, we find

$$\eta_r = -\frac{2\gamma T_0}{R}\delta(r - R). \quad (4.11)$$

Comparison with (4.7) leads to a conclusion that the temperature distribution is exactly equivalent to the distribution of dislocations with Burgers vector $b = \gamma T_0 R$.

As $R \to 0$, the limiting form of a (single) spherical dislocation is a point source with densities (space $\delta(x)$ corresponds to $(2\pi r^2)^{-1}\delta(r)$).

$$\mu_r = \frac{\delta(r)}{2\pi r^2}, \quad \mu_\vartheta = 0; \qquad \eta_r = -\frac{\delta(r)}{\pi r^4}, \quad (4.12)$$

$$\sigma_r = \frac{2\mu}{3\pi(\lambda + 2\mu)}\left[(3\lambda + 2\mu)\frac{\theta(r)}{r^3} + \frac{2}{3}(3\lambda + 4\mu)\frac{\delta(r)}{r^2}\right], \quad (4.13)$$

$$\sigma_\vartheta = -\frac{\mu}{3\pi(\lambda + 2\mu)}\left[(3\lambda + 2\mu)\frac{\theta(r)}{r^3} - \frac{1}{3}(3\lambda - 2\mu)\frac{\delta(r)}{r^2}\right]. \quad (4.14)$$

Now let

$$\eta_r = -a\theta(R - r) \quad (a = \mathrm{const}). \quad (4.15)$$

This incompatibility corresponds, for instance, to a temperature distribution

$$T(r) = T_0\left(1 - \frac{r^2}{R^2}\right)\theta(R - r), \qquad T_0 = \frac{aR^2}{4\gamma}. \quad (4.16)$$

Solution of the equations for the stresses gives

$$\sigma_r = \frac{\mu(3\lambda + 2\mu)}{15(\lambda + 2\mu)}a\left[(5R^2 - 3r^2)\theta(R - r) + \frac{2R^5}{r^3}\theta(r - R)\right]$$
$$\sigma_\vartheta = \frac{\mu(3\lambda + 2\mu)}{15(\lambda + 2\mu)}a\left[(5R^2 - 6r^2)\theta(R - r) - \frac{R^5}{r^3}\theta(r - R)\right]. \quad (4.17)$$

In conclusion, let us consider the distribution of dislocations on the surface of a cylinder of radius $\rho = R$, with Burgers vector b = const oriented along ρ. The corresponding expressions for nonzero components μ and η are in the form

$$\mu_\rho = b\delta(\rho - R), \qquad \eta_z = -\frac{b}{R}\delta(\rho - R), \quad (4.18)$$

$$\sigma_\rho = \frac{2\mu(\lambda + \mu)}{\lambda + 2\mu}\frac{b}{R}\left[\theta(R - \rho) + \frac{R^2}{\rho^2}\theta(\rho - R)\right],$$
$$\sigma_\varphi = \frac{2\mu(\lambda + \mu)}{\lambda + 2\mu}\frac{b}{R}\left[\theta(R - \rho) - \frac{R^2}{\rho^2}\theta(\rho - R)\right]. \quad (4.19)$$

As $R \to 0$ we obtain at the limit a linear source

$$\mu_\rho = \frac{\delta(\rho)}{\pi\rho}, \qquad \eta_z = \frac{2}{\pi}\frac{\delta(\rho)}{\rho^3}, \quad (4.20)$$

$$\sigma_\rho = \frac{\mu(\lambda + \mu)}{\pi(\lambda + 2\mu)}\left[\frac{2\theta(\rho)}{\rho^2} + \frac{\delta(\rho)}{\rho}\right],$$
$$\sigma_\varphi = -\frac{\mu(\lambda + \mu)}{\pi(\lambda + 2\mu)}\left[\frac{2\theta(\rho)}{\rho^2} - \frac{\delta(\rho)}{\rho}\right]. \quad (4.21)$$

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